

Hasil Turnitin Paper Journal FJMS 2018

by Muhammad Jakfar

Submission date: 31-Jul-2018 08:40AM (UTC+0700)

Submission ID: 986446662

File name: MS102050979_1.pdf (150.3K)

Word count: 2661

Character count: 11788

THY-ANGLE IN AN n -NORMED SPACE

Manuharawati and Muhammad Jakfar

Universitas Negeri Surabaya
Kampus Ketintang, Jl. Ketintang
Gedung C8, Surabaya 60231
Indonesia

8

Abstract

In this paper, we introduce the notion of angle and Thy-angle in an n -normed space and prove some fundamental properties and a characterization theorem for Thy-angle in an n -normed space.

1. Introduction

The notion of Thy-angle in normed space has been introduced in [15, 16]. The study on an n -normed space, which is a generalization of normed space, has been done by mathematicians as outlined in [2-11]. By considering those, a notion of Thy-angle in an n -normed space, a generalization of Thy-angle, is already constructed in [15].

On the other hand, in 2008, Thillaigovindan et al. [14] have constructed a notion of angle between two vectors in an n -normed space. However, there are some gaps in Definition 3.1 [14]. The gaps are: (i) if $\{x_2, \dots, x_n\}$ is linearly dependent, then $\|x, x_2, \dots, x_n\|$ or $\|y, x_2, \dots, x_n\|$ is zero, so

$$\|x, x_2, \dots, x_n\| = 0 \quad \text{or} \quad \|y, x_2, \dots, x_n\| = 0$$

Received: April 13, 2017; Revised: May 26, 2017; Accepted: June 20, 2017

2010 Mathematics Subject Classification: 46A99, 46B20, 46S99.

Keywords and phrases: characterization theorem, n -norm space, angle, Thy-angle.

$\theta(x, y; x_2, \dots, x_n)$ in [14] is not defined on X ; (ii) if x or y is in $\text{span}\{x_2, \dots, x_n\}$, then $\theta(x, y; x_2, \dots, x_n)$ is not defined on X . By the existence of gaps, a notion of angle between two vectors in an n -normed space is developed which is an improvement of Definition 3.1 [14].

In this paper, the notion of Thy-angle in an n -normed space along with the new notion of angle between two vectors in an n -normed space, is used for proving a characterization theorem for Thy-angle in an n -normed space.

2. Preliminaries

This section provides a collection of basic definitions and results needed for the development of the new concepts.

Definition 2.1 [8]. Let n be a natural number ($n \in \mathbb{N}$) and X be a real vector space having $\dim(X) \geq n$. A real valued function $\|\cdot, \dots, \cdot\|$ on X^n is said to be an n -norm on X if for any $x_1, x_2, \dots, x_n \in X$, the following four properties hold:

- (i) $\|x_1, \dots, x_n\| = 0$ if and only if $\{x_1, x_2, \dots, x_n\}$ is linear dependent on X ,
- (ii) $\|x_1, \dots, x_n\|$ is invariant under any permutation,
- (iii) $\|\alpha x_1, \dots, x_n\| = |\alpha| \|x_1, \dots, x_n\|$ for any $\alpha \in \mathbb{R}$,
- (iv) $\|x + y, \dots, x_n\| \leq \|x, \dots, x_n\| + \|y, \dots, x_n\|$.

The pair $(X, \|\cdot, \dots, \cdot\|)$ is called an n -normed space on X .

Definition 2.2 [1]. Let $n \in \mathbb{N}$ and X be a real vector space with $\dim(X) \geq n$. A real valued function $\langle \cdot, \cdot | \cdot, \dots, \cdot \rangle$ on X^{n+1} is said to be an n -inner product on X if for any $x_1, x_2, \dots, x_n \in X$ satisfying the following conditions:

- (i) $\langle x, x | x_2, x_3, \dots, x_n \rangle \geq 0$,
- (ii) $\langle x, x | x_2, x_3, \dots, x_n \rangle = 0$ if and only if $\{x, x_2, \dots, x_n\}$ is linear dependent,
- (iii) $\langle x, y | x_2, x_3, \dots, x_n \rangle = \langle y, x | x_2, x_3, \dots, x_n \rangle$,
- (iv) $\langle x, y | x_2, x_3, \dots, x_n \rangle$ is invariant under any permutation of $x_2, \dots, x_n$,
- (v) $\langle x, x | x_2, x_3, \dots, x_n \rangle = \langle x_2, x_2 | x, x_3, \dots, x_n \rangle$,
- (vi) $\langle \alpha x, x | x_2, x_3, \dots, x_n \rangle = \alpha \langle x, x | x_2, x_3, \dots, x_n \rangle$ for any $\alpha \in \mathbb{R}$,
- (vii) $\langle x + x', y | x_2, x_3, \dots, x_n \rangle = \langle x, y | x_2, x_3, \dots, x_n \rangle + \langle x', y | x_2, x_3, \dots, x_n \rangle$.

The pair $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$ is called an n -inner product space on X .

Theorem 2.3 [1]. If an n -inner product space on X is given, then

- (i) $\|x_1, x_2, \dots, x_n\| = \sqrt{\langle x_1, x_1 | x_2, x_3, \dots, x_n \rangle}$ defines an n -norm on X ,
- (ii) the extension of Cauchy-Buniakowski inequality:
- $$\langle x, y | x_2, x_3, \dots, x_n \rangle \leq \sqrt{\langle x, x | x_2, x_3, \dots, x_n \rangle} \cdot \sqrt{\langle y, y | x_2, x_3, \dots, x_n \rangle}$$

also holds.

Theorem 2.4 [12]. If an n -inner product space on X is given, then

- $\|x_1, x_2, \dots, x_n\| = \sqrt{\langle x_1, x_1 | x_2, x_3, \dots, x_n \rangle}$ defines an n -norm on X for which
- $$\langle x, y | x_2, x_3, \dots, x_n \rangle = \frac{1}{4} \{ \|x + y, x_2, \dots, x_n\|^2 - \|x - y, x_2, \dots, x_n\|^2 \},$$
- $$\langle x, y | x_2, x_3, \dots, x_n \rangle = \frac{1}{2} \{ \|x, x_2, \dots, x_n\|^2 + \|y, x_2, \dots, x_n\|^2 - \|x - y, x_2, \dots, x_n\|^2 \}$$

and

$$\begin{aligned} & \|x + y, x_2, \dots, x_n\|^2 + \|x - y, x_2, \dots, x_n\|^2 \\ &= 2\{\|x, x_2, \dots, x_n\|^2 + \|y, x_2, \dots, x_n\|^2\}. \end{aligned}$$

Definition 2.5 [15]. Let $(X, \|\cdot\|)$ be a normed space and $x, y \in X$. A Thy-angle between x and y , denoted by $A_{Thy}(x, y)$, is defined as

$$A_{Thy}(x, y) = \arccos \left[\frac{1}{4} \left(\left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|^2 - \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 \right) \right].$$

Theorem 2.6 [15]. $A_{Thy}(x, y)$ lies between 0 and π .

3. Discussion

3.1. Thy-angle in an n -normed space

In this section, we will use X is a real vector space with $\dim(X) \geq n$.

Definition 3.1.1. Let $(X, \|\cdot, \dots, \cdot\|)$ be an n -normed space and $x, y, x_1, x_2, x_3, \dots, x_n \in X$, $\{x_1, x_2, x_3, \dots, x_n\}$ be a linearly independent set. A Thy-angle between x and y with respect to $x_1, x_2, x_3, \dots, x_n$, denoted by $A_{Thy}(x, y, x_1, \dots, x_n)$, is defined as

$$\begin{aligned} & A_{Thy}(x, y, x_1, \dots, x_n) \\ &= \arccos \left[\frac{1}{4} \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left(\left\| \frac{x}{\varphi} + \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 - \left\| \frac{x}{\varphi} - \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right) \right] \end{aligned}$$

with

$$\varphi = \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|x, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}$$

and

$$\Psi = \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|y, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}.$$

Theorem 3.1.2. Let $(X, \|\cdot, \dots, \cdot\|)$ be an n -normed space and $x, y, x_1, x_2, x_3, \dots, x_n \in X$. If $\{x_1, x_2, \dots, x_n\}$ is a linearly independent set, then

- (i) $A_{Thy}(x, y, x_1, \dots, x_n) = A_{Thy}(y, x, x_1, \dots, x_n)$.
- (ii) $A_{Thy}(\alpha x, \beta y, x_1, \dots, x_n) = A_{Thy}(x, y, x_1, \dots, x_n)$, for $\alpha, \beta \in \mathbb{R}$ with $\beta > 0$.
- (iii) $A_{Thy}(\alpha x, \beta y, x_1, \dots, x_n) = \pi - A_{Thy}(x, y, x_1, \dots, x_n)$, for $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta < 0$.
- (iv) $A_{Thy}(x, y, x_1, \dots, x_n) = 0$ if and only if x and y are two vectors of the same direction.
- (v) $A_{Thy}(x, y, x_1, \dots, x_n) = \pi$ if and only if x and y are in opposite direction.
- (vi) If $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$ is an n -inner product space, then

$$A_{Thy}(x, y, x_1, \dots, x_n) = \arccos \left[\frac{\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \langle x, y | x_{i_2}, \dots, x_{i_n} \rangle}{\Phi \cdot \Psi} \right]$$

with

$$\Phi = \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|x, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}$$

and

$$\Psi = \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|y, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}.$$

(vii) If $\varphi = \psi$, then

$$4\varphi^2 \cos A_{Thy}(x, y; x_1, \dots, x_n) \\ = \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} (\|x + y, x_{i_2}, \dots, x_{i_n}\|^2 - \|x - y, x_{i_2}, \dots, x_{i_n}\|^2)$$

with

$$\varphi = \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|x, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}$$

and

$$\psi = \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|y, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}.$$

Proof. For (i) to (v), it follows immediately from Definition 3.1.1.

²⁸ Since $\{x_1, x_2, \dots, x_n\}$ is a linearly independent set, by applying Theorem

2.4, we have

$$\left\langle \frac{x}{\varphi}, \frac{y}{\psi} \middle| x_{i_2}, \dots, x_{i_n} \right\rangle = \frac{1}{4} \left\{ \left\| \frac{x}{\varphi} + \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 - \left\| \frac{x}{\varphi} - \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right\}.$$

So,

$$\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left\langle \frac{x}{\varphi}, \frac{y}{\psi} \middle| x_{i_2}, \dots, x_{i_n} \right\rangle = \cos A_{Thy}(x, y; x_1, \dots, x_n).$$

This proves (vi).

To prove (vii) we proceed as follows.

Since $\varphi = \psi$,

$$\begin{aligned} & \frac{1}{4} \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left(\left\| \frac{x}{\varphi} + \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 - \left\| \frac{x}{\varphi} - \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right) \\ &= \frac{1}{4\varphi^2} \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} (\|x + y, x_{i_2}, \dots, x_{i_n}\|^2 - \|x - y, x_{i_2}, \dots, x_{i_n}\|^2). \end{aligned}$$

Consequently,

$$\begin{aligned} & 4\varphi^2 \cos A_{Thy}(x, y, x_1, \dots, x_n) \\ &= \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} (\|x + y, x_{i_2}, \dots, x_{i_n}\|^2 - \|x - y, x_{i_2}, \dots, x_{i_n}\|^2). \quad \square \end{aligned}$$

Theorem 3.1.3. Let $x, y, x_1, x_2, \dots, x_n \in X$. If $\{x_1, x_2, \dots, x_n\}$ is a linearly independent set, then for any real number $t \neq 0$, we have

$$\begin{aligned} & \lim_{t \rightarrow \infty} \left(\frac{tx + y}{\left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|tx + y, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}} \right) \\ &= \frac{x}{\left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|x, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}} \end{aligned}$$

and

$$\begin{aligned} & \lim_{t \rightarrow -\infty} \left(\frac{tx + y}{\left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|tx + y, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}} \right) \\ &= \frac{-x}{\left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|x, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}}. \end{aligned}$$

Proof. Let $|t|$ be sufficiently large with $tx + y \neq 0$.

If $t > 0$, then

$$\begin{aligned} & \lim_{t \rightarrow \infty} \left(\frac{tx + y}{\left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|tx + y, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}} \right) \\ &= \lim_{t \rightarrow \infty} \left(\frac{x + y/t}{\left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|x + y/t, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}} \right) \\ &= \frac{x}{\left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|x, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}}. \end{aligned}$$

If $t < 0$, then

$$\begin{aligned} & \frac{tx + y}{\left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|tx + y, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}} \\ &= \frac{-x + y/k}{\left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|-x + y/k, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}}, \end{aligned}$$

where $t = -k$, $k > 0$ and

$$\lim_{t \rightarrow -\infty} \left(\frac{tx + y}{\left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|tx + y, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}} \right)$$

$$= - \frac{x}{\left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|x, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}}.$$

□

Theorem 3.1.4. $A_{Thy}(x, y, x_1, \dots, x_n)$ lies between 0 and π .

Proof. Let

$$\varphi = \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|x, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}$$

and

$$\psi = \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|y, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}.$$

Since

$$\begin{aligned} & \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left\| \frac{x}{\varphi} + \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \\ & \leq \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left(\left\| \frac{x}{\varphi}, x_{i_2}, \dots, x_{i_n} \right\| + \left\| \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\| \right)^2 \\ & \leq \left[\left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left\| \frac{x}{\varphi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right)^{\frac{1}{2}} \right. \\ & \quad \left. + \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left\| \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right)^{\frac{1}{2}} \right]^2 = 4 \end{aligned}$$

and

$$\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left(\left\| \frac{x}{\varphi} - \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right)$$

$$\leq \left[\left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left\| \frac{x}{\phi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right)^{\frac{1}{2}} + \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left\| \frac{y}{\phi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right)^{\frac{1}{2}} \right]^2 = 4,$$

we have

$$\begin{aligned} -4 &\leq \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left(\left\| \frac{x}{\phi} + \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 - \left\| \frac{x}{\phi} - \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right) \\ &\leq 4. \end{aligned}$$

Therefore $A_{Thy}(x, y, x_1, \dots, x_n)$ lies between 0 and π . $\square$

Theorem 3.1.5. Let $x, y, x_1, x_2, x_3, \dots, x_n \in V$. If $\{x_1, x_2, \dots, x_n\}$ is a linearly independent set, then for each $\alpha \in (0, \pi)$, there exists a real number a such that

$$A_{Thy}(x, \alpha x + y, x_1, \dots, x_n) = \alpha.$$

Proof. By using Theorem 3.1.5, it is sufficient to see that for any $k \in (-1, 1)$ there exists a number a such that

$$\frac{1}{4} \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left(\left\| \frac{x}{\phi} + \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 - \left\| \frac{x}{\phi} - \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right) = k.$$

By using Theorem 3.1.4, we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \left(\frac{1}{4} \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left(\left\| \frac{x}{\phi} + \frac{tx + y}{\omega}, x_{i_2}, \dots, x_{i_n} \right\|^2 - \left\| \frac{x}{\phi} - \frac{tx + y}{\omega}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right) \right) &= 1 \end{aligned}$$

and

$$\lim_{t \rightarrow -\infty} \left(\frac{1}{4} \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left(\left\| \frac{x}{\varphi} + \frac{tx + y}{\omega}, x_{i_2}, \dots, x_{i_n} \right\|^2 - \left\| \frac{x}{\varphi} - \frac{tx + y}{\omega}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right) \right) = -1$$

with

$$\varphi = \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|x, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}},$$

$$\psi = \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|y, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}},$$

and

$$\omega = \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|tx + y, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}.$$

Since

$$g(t) = \frac{1}{4} \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left(\left\| \frac{x}{\varphi} + \frac{tx + y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 - \left\| \frac{x}{\varphi} - \frac{tx + y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right)$$

is a continuous function at t , the result follows. $\square$

3.2. The angle between two vectors

The following definition is a revised version of Definition 3.1 [14].

Definition 3.2.1. Let $(X, \|\cdot, \dots, \cdot\|)$ be an n -normed space and $x, y, x_1, x_2, x_3, \dots, x_n \in X$, $\{x_1, x_2, x_3, \dots, x_n\}$ be a linearly independent set. The angle between x and y with respect to $x_1, x_2, x_3, \dots, x_n$, denoted by $A_*(x, y, x_1, \dots, x_n)$, is defined as

$$A_*(x, y, x_1, \dots, x_n) = \arccos \left[\frac{1}{2} \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left(2 - \left\| \frac{x}{\varphi} - \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right) \right]$$

with

$$\varphi = \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|x, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}$$

and

$$\psi = \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|y, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}.$$

The relation between A_{Thy} and A_* which expressed in the following theorem, is called as a *characterization theorem* for Thy-angle.

Theorem 3.2.2 (Characterization theorem). Let $(X, \|\cdot, \dots, \cdot\|)$ be an n -normed space. The pair $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$ is an n -inner product space if and only if

$$A_{Thy}(x, y, x_1, \dots, x_n) = A_*(x, y, x_1, \dots, x_n)$$

for any $x, y, x_1, x_2, \dots, x_n \in X$, $\{x_1, x_2, \dots, x_n\}$ is a linearly independent set.

Proof. By using (vi) of Theorem 3.1.2, we have

$$A_{Thy}(x, y, x_2, \dots, x_n) = \arccos \left[\frac{\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \langle x, y | x_{i_2}, \dots, x_{i_n} \rangle}{\varphi \cdot \psi} \right].$$

We also have

$$A_{**}(x, y, x_2, \dots, x_n) = \arccos \left[\frac{\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \langle x, y | x_{i_2}, \dots, x_{i_n} \rangle}{\varphi \cdot \psi} \right]$$

with

$$\varphi = \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|x, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}$$

and

$$\psi = \left(\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|y, x_{i_2}, \dots, x_{i_n}\|^2 \right)^{\frac{1}{2}}.$$

By applying Theorem 2.4 to $\frac{x}{\varphi}$ and $\frac{y}{\psi}$, for any $x, y, x_1, \dots, x_n \in X$,

$\{x_1, x_2, \dots, x_n\}$ is a linearly independent set, we have

$$\begin{aligned} & \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \left\langle \frac{x}{\varphi}, \frac{y}{\psi} \middle| x_{i_2}, \dots, x_{i_n} \right\rangle \\ &= \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \frac{1}{2} \left\{ \left\| \frac{x}{\varphi}, x_{i_2}, \dots, x_{i_n} \right\|^2 + \left\| \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right. \\ & \quad \left. - \left\| \frac{x}{\varphi} - \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right\} \end{aligned}$$

$$\begin{aligned}
&= \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \frac{1}{2} \left\{ 2 - \left\| \frac{x}{\varphi} - \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right\} \\
&= \cos A_*(x, y, x_1, \dots, x_n).
\end{aligned}$$

So, $A_{Thy}(x, y, x_1, \dots, x_n) = A_*(x, y, x_1, \dots, x_n)$.

Conversely, let

$$A_{Thy}(x, y, x_1, \dots, x_n) = A_*(x, y, x_1, \dots, x_n)$$

for any $x, y, x_1, x_2, \dots, x_n \in X$, with $\{x_2, x_3, \dots, x_n\}$ be a linear independent set. By applying Theorem 2.4, it is enough to see that parallelogram law holds for $x, y, x_1, x_2, \dots, x_n \in X$, $\{x_1, x_2, \dots, x_n\}$ is a linear independent set.

If we take $\varphi = \psi = 1$, then

$$\begin{aligned}
&\cos A_{Thy}(x, y, x_1, \dots, x_n) \\
&= \frac{1}{4} \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} (\|x + y, x_{i_2}, \dots, x_{i_n}\|^2 - \|x - y, x_{i_2}, \dots, x_{i_n}\|^2),
\end{aligned}$$

and

$$\cos A_*(x, y, x_2, \dots, x_n) = \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \frac{1}{2} \left\{ 2 - \left\| \frac{x}{\varphi} - \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right\}.$$

Therefore,

$$\begin{aligned}
&\frac{1}{4} \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} (\|x + y, x_{i_2}, \dots, x_{i_n}\|^2 - \|x - y, x_{i_2}, \dots, x_{i_n}\|^2) \\
&= \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \frac{1}{2} \left\{ 2 - \left\| \frac{x}{\varphi} - \frac{y}{\psi}, x_{i_2}, \dots, x_{i_n} \right\|^2 \right\}.
\end{aligned}$$

Since $x, y, x_1, x_2, \dots, x_n \in X$ are arbitrary,

$$\|x - y, x_2, \dots, x_n\|^2 - \|x + y, x_2, \dots, x_n\|^2 = 4 - 2\|x - y, x_2, \dots, x_n\|^2.$$

Hence

$$\|x - y, x_2, \dots, x_n\|^2 + \|x + y, x_2, \dots, x_n\|^2 = 4.$$

It shows that $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$ is an n -inner product space. $\square$

References

- [1] Y. J. Cho, M. Matic and J. Pacarie, Inequalities of Hlawka's type in n -inner product spaces, Comm. Korean Math. Soc. 17(4) (2002), 583-592.
- [2] Y. J. Cho, P. C. S. Lin, S. S. Kim and A. Misiak, Theory of 2-inner Product Spaces, Nova Science Publisher, Inc., Huntington, NY, 2001.
- [3] C. Diminnie, S. Gähler and A. White, 2-inner product spaces, Demonstratio Math. 6 (1973), 525-536.
- [4] C. Diminnie, S. Gähler and A. White, 2-inner product spaces. II, Demonstratio Math. 10(1) (1977), 169-188.
- [5] S. Gähler, Lineare 2-normierte Räume, Math. Nachr. 28 (1965), 1-43.
- [6] S. Gähler, Untersuchungen über verallgemeinerte m -metrische Räume. I, Math. Nachr. 40 (1969), 165-189.
- [7] S. Gähler, Untersuchungen über verallgemeinerte m -metrische Räume. II, Math. Nachr. 40 (1969), 229-264.
- [8] H. Gunawan and Mashadi, On n -normed spaces, Int. J. Math. Sci. 27 (2001), 631-639.
- [9] H. Gunawan, Mashadi, S. Gemawati, Nursupiamin and I. Sihwaningrum, Orthogonality in 2-normed spaces revisited, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. 17 (2006), 76-83.
- [10] H. Mazaheri and S. G. Nezhad, Some results on b -orthogonality in 2-normed linear spaces, Int. J. Math. Anal. 1 (2007), 681-687.
- [11] A. Misiak, n -inner product spaces, Math. Nachr. 140 (1989), 299-319.
- [12] A. Misiak, Orthogonality and orthonormality in n -inner product spaces, Math. Nachr. 143 (1989), 249-261.

- [13] B. Surender Reddy, Intuitionistic fuzzy 2-norm, *Int. J. Math. Anal.* (Ruse) 5(13-16) (2011), 651-659.
- [14] N. Thillaigovindan, S. Vijayabalaji and S. Anita Shanthi, Angles in n -normed linear space, *Proceedings of National Seminar on Mathematical Methods for Research*, A. V. C. College, 2008, pp. 61-68.
- [15] V. Thürey, Angles and polar coordinates in real normed spaces, *arXiv:0902.2731v2*.
- [16] V. Thürey, Angles and a classification of normed spaces, *Ann. Funct. Anal.* 4(1) (2013), 114-137.

Hasil Turnitin Paper Journal FJMS 2018

ORIGINALITY REPORT

19%

SIMILARITY INDEX

12%

INTERNET SOURCES

19%

PUBLICATIONS

%

STUDENT PAPERS

PRIMARY SOURCES

1

"Notations and Basics", Solid Mechanics and Its Applications, 2007

Publication

2%

2

inf.central.ucv.ro

Internet Source

2%

3

F Kamareddine. "Bridging de Bruijn indices and variable names in explicit substitutions calculi", Logic Journal of IGPL, 11/01/1998

Publication

1%

4

Mohammad Ashraf, Mohammad Aslam Siddeeqe, Nazia Parveen. " On semigroup ideals and -derivations in near-rings ", Journal of Taibah University for Science, 2018

Publication

1%

5

www.mathem.pub.ro

Internet Source

1%

6

www.kurims.kyoto-u.ac.jp

Internet Source

1%

7

Dutta, H.. "On sequence spaces with elements

1%

in a sequence of real linear n -normed spaces",
Applied Mathematics Letters, 201009

Publication

8	fs.gallup.unm.edu Internet Source	1 %
9	jipam.vu.edu.au Internet Source	1 %
10	"Advances in Neural Networks - ISNN 2006", Springer Nature America, Inc, 2006 Publication	1 %
11	Aleksander Misiak. "Orthogonality and Orthonormality in n -Inner Product Spaces", Mathematische Nachrichten, 1989 Publication	1 %
12	preview- advancesindifferenceequations.springeropen.com Internet Source	1 %
13	Aleksander Misiak. " n -Inner Product Spaces", Mathematische Nachrichten, 1989 Publication	1 %
14	www.researchgate.net Internet Source	1 %
15	S.K. Sharma, Ayhan Esi. "Some I -convergent sequence spaces defined by using sequence of moduli and n -normed space", Journal of the Egyptian Mathematical Society, 2013	1 %

16

Busneag, D.. "On extensions of pseudo-valuations on Hilbert algebras", Discrete Mathematics, 20030228

Publication

<1 %

17

Timothy J. Healy. "The Capacity Region of a Spread Spectrum Multiple-Access Channel", MILCOM 1982 - IEEE Military Communications Conference - Progress in Spread Spectrum Communications, 1982

Publication

<1 %

18

ijopaasat.in

Internet Source

<1 %

19

Pei-Hen Tsai, Yung-Yu Chuang. "Target-Driven Moire Pattern Synthesis by Phase Modulation", 2013 IEEE International Conference on Computer Vision, 2013

Publication

<1 %

20

Lael, Fatemeh Nourouzi, Kourosh. "Compact operators defined on 2-normed and 2-probabilistic normed spaces.(Research Article) (Report)", Mathematical Problems in Engineering, Annual 2009 Issue

Publication

<1 %

21

Bhaba R. Sarker. "Optimum manpower models for a production system with varying production rates", European Journal of Operational

<1 %

22

Chun-Gil Park. "Generalized Hyers–Ulam–Rassias stability of n -sesquilinear-quadratic mappings on Banach modules over C^* -algebras", Journal of Computational and Applied Mathematics, 2005

Publication

<1 %

23

arxiv.org

Internet Source

<1 %

24

Park, C.G.. "Modified Trif's functional equations in Banach modules over a C^* -algebra and approximate algebra homomorphisms", Journal of Mathematical Analysis and Applications, 20030201

Publication

<1 %

25

alg-d.com

Internet Source

<1 %

26

elib.mi.sanu.ac.rs

Internet Source

<1 %

27

www.cirm.univ-mrs.fr

Internet Source

<1 %

28

Mursaleen, M.. "Intuitionistic fuzzy 2-normed space and some related concepts", Chaos, Solitons and Fractals, 20091015

Publication

<1 %

29

hackersdelight.org

Internet Source

<1 %

30

dml.cz

Internet Source

<1 %

31

Menegatto, V.A.. "Strictly positive definite kernels on subsets of the complex plane", Computers and Mathematics with Applications, 200604

Publication

<1 %

32

Irina Georgescu. "Arrow's Axiom and Full Rationality for Fuzzy Choice Functions", Social Choice and Welfare, 01/05/2007

Publication

<1 %

Exclude quotes

Off

Exclude matches

< 3 words

Exclude bibliography

On

Hasil Turnitin Paper Journal FJMS 2018

GRADEMARK REPORT

FINAL GRADE

/0

GENERAL COMMENTS

Instructor

PAGE 1

PAGE 2

PAGE 3

PAGE 4

PAGE 5

PAGE 6

PAGE 7

PAGE 8

PAGE 9

PAGE 10

PAGE 11

PAGE 12

PAGE 13

PAGE 14

PAGE 15

PAGE 16